Renormalization of tracer turbulence leading to fractional differential equations

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For many years quasilinear renormalization has been applied to numerous problems in turbulent transport. This scheme relies on the localization hypothesis to derive a linear transport equation from a simplified stochastic description of the underlying microscopic dynamics. However, use of the localization hypothesis narrows the range of transport behaviors that can be captured by the renormalized equations. In this paper, we construct a renormalization procedure that manages to avoid the localization hypothesis completely and produces renormalized transport equations, expressed in terms of fractional differential operators, that exhibit much more of the transport phenomenology observed in nature. This technique provides a first step toward establishing a rigorous link between the microscopic physics of turbulence and the fractional transport models proposed phenomenologically for a wide variety of turbulent systems such as neutral fluids or plasmas.

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I. INTRODUCTION

Phenomenological models based on continuous-time random walks (CTRWs) [1] or the related concept of fractional differential equations (FDEs) [2–5] have recently become commonplace in many fields in which turbulent transport is present. These include numerous problems in neutral fluids, solar and atmospheric turbulence, combustion, and many other areas where anomalous scalings are observed (see Refs. [6–12] and references therein). In the last few years, they have also been proposed as potential candidates to describe radial turbulent transport in magnetically confined plasmas [13–18]. Previously, effective transport coefficients (diffusivities, conductivities, and others) had been traditionally estimated first from theory (by means of quasilinear or more elaborate renormalization schemes [19–22]) or numerical simulations [23–26]. Then, they were built into classical diffusive equations (i.e., for mass, momentum, or energy) and contrasted (and improved) against experimental data [27,28]. However, the validity of this standard paradigm based on a transport matrix concept requires the existence of well-defined characteristic time and length scales [29,30], a situation that experimental evidence suggests may not apply to these plasmas [31–38].

In fact, in many simulations of turbulent plasmas, it has been observed instead that the density of tracer particles radially transported across the system satisfies fractional transport equations of the form [13–16,39],

$$ {}_0^C D_t^{\beta} n_0(x,t) = D_{[\beta,\alpha]} \frac{\partial^{\alpha}}{\partial |x|^{\alpha}} n_0(x,t), \quad (1) $$

with $\beta < 1$ and $\alpha < 2$. This behavior is observed across a range of temporal and spatial scales spanning from a few times the minimum ones set by the simulation resolution to the maximum ones set by the simulation duration and system size. Here, ${}_0^C D_t^{\beta}$ is the Caputo temporal fractional differential operator (FDO) of order β, with $\beta \in (0,1]$. On the other hand, $\partial^{\alpha}/\partial|x|^{\alpha}$ is the Riesz symmetric FDO of order α, with $\alpha \in (0,2]$ (for an accessible introduction to the fractional differential operators used in this paper see, for instance, Ref. [40]). These operators reduce to the usual derivatives only when $\beta \to 1^-, \alpha \to 2^-$ (i.e., the diffusive limit). Then, the standard diffusive equation is recovered. $D_{[\beta,\alpha]}$ is an effective "fractional diffusivity."

FDEs like Eq. (1) are non-Markovian and non-Gaussian in nature except when $\alpha=2$ and $\beta=1$ [9]. One of their features is that the average tracer displacement satisfies $\langle |\Delta x|^s \rangle \propto t^{sH}$ with transport exponent

$$ H = \beta/\alpha \quad (2) $$

for any $s>0$ for which this average converges. Note that superdiffusion ($H>1/2$), subdiffusion ($H<1/2$), and diffusion ($H=1/2$) are all contained within Eq. (1). Therefore, FDEs will in general be better suited than a standard diffusive equation to capture transport dynamics whenever $H \neq 1/2$ is observed. Note that subdiffusion always requires $\beta<1$ for $\alpha=2$. But, for $\alpha<2$, superdiffusion ($2\beta>\alpha$) and even diffusion ($2\beta=\alpha$) can also take place when $\beta<1$. These situations are indeed found in some systems. For instance, the aforementioned plasma simulations point to $\beta<1$, $\alpha<2$, and $2\beta>\alpha$ as the range of parameter values consistent with the turbulent regimes of interest in tokamaks, in which concepts like self-organized criticality are thought to play an important role [41–45].

No formal procedure is, however, known that allows one to derive Eq. (1) formally from some reasonable description of the microscopic dynamics of the turbulent flow transporting the tracers [22], except for a very reduced group of special cases. The fact that the aforementioned numerical simulations provide with a "numerical derivation" suggests that a

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procedure should indeed exist. In this paper, we present one such scheme starting from the simplest microscopic description one can think of, the equation of motion for a single tracer:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V}(\mathbf{r},t), \tag{3}$$

where $\mathbf{V}(\mathbf{r},t)$ is an incompressible n_D-dimensional turbulent flow with prescribed statistics (say, an $\mathbf{E}\times\mathbf{B}$ flow as in Refs. [13,14]). The "kinetic" equation associated with Eq. (3) is the continuity equation for the tracer density,

$$\frac{\partial n}{\partial t} + \mathbf{V}\cdot\nabla n = 0. \tag{4}$$

We use the term "renormalization" for any transformation procedure that turns Eq. (4) into a linear transport equation of the class defined by Eq. (1), with some diffusivity set by the flow statistics. The simplest scheme, leading to the classical diffusive equation, is quasilinear renormalization (QLR) [22,30]. It requires, however, an additional assumption: the so-called localization hypothesis. The result is the well-known Lagrangian velocity correlation formalism, which has been applied to many turbulent problems in physics and engineering [22,30].

We will show that the localization hypothesis is precisely the reason why traditional (and extended versions of) QLR fails to produce renormalized linear equations like Eq. (1) for $\alpha<2$. To circumvent this limitation, we have constructed a different renormalization technique that does not assume the localization hypothesis. As a result, it allows the direct derivation of a much larger family of renormalized transport equations that contains Eq. (1) for all $\beta\in(0,1]$ and $\alpha\in(0,2]$. But, in addition, it also produces FDEs that cannot be related to any microscopic separable CTRW based on stable (i.e., Lévy) distributions.

The paper is thus organized as follows. In Sec. II, the traditional quasilinear renormalization of Eq. (4) will be reviewed. Several subtle points of QLR, which will play a role in later sections, will be discussed in detail. In Sec. III, we proceed to describe our renormalization technique that proceeds without using the localization hypothesis. The Lagrangian correlation concept thus ceases to be important. In Sec. IV, as an example of application, we carry out explicitly the renormalization of several one-dimensional self-similar flows. Finally, in Sec. V, some conclusions will be drawn.

II. RENORMALIZATION OF EQ. (4) VIA THE LAGRANGIAN CORRELATION FORMALISM

The objective of quasilinear renormalization techniques and their extensions is to turn Eq. (4) into a diffusive transport equation with some renormalized diffusivity. Typically, one proceeds by separating ensemble-averaged $\langle\ \rangle$ and fluctuating $(\tilde{\ })$ parts of density and flow,

$$\frac{\partial n_0}{\partial t} + \mathbf{V}_0\cdot\nabla n_0 = -\langle\tilde{\mathbf{V}}\cdot\nabla\tilde{n}\rangle, \tag{5}$$

$$\frac{\partial \tilde{n}}{\partial t} + \mathbf{V}_0\cdot\nabla\tilde{n} + \tilde{\mathbf{V}}\cdot\nabla\tilde{n} = -\tilde{\mathbf{V}}\cdot\nabla n_0 + \langle\tilde{\mathbf{V}}\cdot\nabla\tilde{n}\rangle, \tag{6}$$

where the ensemble is taken over multiple realizations of the flow $\mathbf{V}(\mathbf{r},t)$. Note that we use the notation $\langle A\rangle\equiv A_0$. Standard QLR proceeds then by neglecting second-order terms only in Eq. (6). Then it solves for $\tilde{n}$ in terms of $\mathbf{V}_0$ and ∇n_0 and uses the result in Eq. (5), which becomes an advection-diffusion equation with a renormalized eddy diffusivity [29]. It is, however, not necessary to neglect the second-order terms: $\tilde{\mathbf{V}}\cdot\nabla\tilde{n}$ can be kept from the start and $\langle\tilde{\mathbf{V}}\cdot\nabla\tilde{n}\rangle$ can be accounted for afterward by iteration [22]. We will follow this path in what follows [denoting it as extended QLR (EQLR)] because it has the advantage of dealing with the exact propagator (i.e., the one prescribed by the full flow) instead of the one associated with the average flow $\mathbf{V}_0$. This feature is essential if we do not want to deal with the asymmetries that a nonzero mean flow may impose on the dynamics. This will be the case studied in the current paper, and we will thus set $\mathbf{V}_0=0$ (note, however, that if the mean flow is uniform, a simple change of coordinates will also bring us to the zero-flow situation). Then, we continue by writing the equation satisfied by the propagator associated with the full flow [assuming $G(\mathbf{r},0)=0$]:

$$\frac{\partial G}{\partial t} + \tilde{\mathbf{V}}\cdot\nabla G = \delta(\mathbf{r}-\mathbf{r}',t-t'), \tag{7}$$

whose formal solution is (for $t>t'$),

$$G(\mathbf{r}-\mathbf{r}',t-t') = \delta(\mathbf{r}'-\mathbf{R}(t'|\mathbf{r},t)). \tag{8}$$

The characteristic $\mathbf{R}(t'|\mathbf{r},t)$ results from solving backward in time (from t to t') the microscopic equation of motion [Eq. (3)], which we repeat here for convenience:

$$\frac{d\mathbf{R}}{d\tau} = \tilde{\mathbf{V}}(\mathbf{R},\tau), \quad \mathbf{R}(t)=\mathbf{r}. \tag{9}$$

Writing now $\tilde{n}(\mathbf{r}',t')$ in terms of the propagator (Eq. (8)) and inserting the result in Eq. (5), we obtain:

$$\frac{\partial n_0}{\partial t} = \nabla\cdot\left(\int_0^t dt'\langle\tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{R}(t'|\mathbf{r},t),t') \times \nabla n_0(\mathbf{R}(t'|\mathbf{r},t),t')\rangle\right). \tag{10}$$

To get to this result, one simply uses the fact that the lowest-order contribution to $\tilde{n}$ obtained from iterating $\langle\tilde{\mathbf{V}}\cdot\nabla\tilde{n}\rangle$ yields, when inserted in Eq. (5), a term proportional to $\langle\tilde{\mathbf{V}}\rangle$ and thus vanishes [8].

It is worth noting that, in contrast to the standard QLR result obtained by neglecting the second-order terms in the fluctuation equation [29], ∇n_0 depends now explicitly on the flow through the characteristic $\mathbf{R}(t'|\mathbf{r},t)$ and thus cannot be taken out of the ensemble average [22]. This complicates the renormalization procedure. The traditional shortcut [22] is to assume the validity of the locality hypothesis $\nabla n_0(\mathbf{R}(t'|\mathbf{r},t),t')\simeq\nabla n_0(\mathbf{r},t')$, which turns Eq. (10) into

$$\frac{\partial n_0}{\partial t} \simeq \nabla \cdot \left(\int_0^t dt' \mathbf{C}^L(t',t) \nabla n_0(\mathbf{r},t') \right). \quad (11)$$

In it, the Lagrangian velocity correlation matrix has been defined as

$$\mathbf{C}^L(t',t) = \int d\mathbf{r}' \left\langle \widetilde{\mathbf{V}}(\mathbf{r},t) \widetilde{\mathbf{V}}(\mathbf{R}(t'|\mathbf{r},t),t') \right\rangle. \quad (12)$$

To conclude the renormalization, one must propose some physically based ansatz for the Lagrangian correlation matrix. Usually, the first step is to assume a homogeneous and isotropic flow, which converts $\mathbf{C}^L(t',t)$ into $\mathbf{C}^L(t-t')$. Then, the final renormalized equation depends on the functional form chosen for $\mathbf{C}^L(t-t')$ [46].

A. Diffusive case

The usual diffusive equation is obtained when assuming that the Lagrangian correlation decays exponentially as $\mathbf{C}^L_{ij}(\tau) = V_c^2 \exp(-\tau/\tau_c)\delta_{ij}$, where V_c is some characteristic velocity. Physically, this choice implies that the random destructive superposition of the fluctuations causes the average flow to "forget" its past very quickly, typically, beyond the characteristic time scale τ_c. Equation (11) then becomes [6,29]

$$\frac{\partial n_0}{\partial t} \simeq D \nabla^2 n_0(\mathbf{r},t); \quad (13)$$

that is, the usual diffusive equation with $D \sim (V_c^2 \tau_c)$.

B. Superdiffusive case

Another possible choice is to assume an algebraically decaying Lagrangian correlation with a positive tail:

$$\mathbf{C}^L_{ij}(\tau) \sim V_c^2 (1-\eta) \left(\frac{\tau}{\tau_c} \right)^{-\eta} \delta_{ij}, \quad \eta \in (0,1). \quad (14)$$

The prefactor $(1-\eta)$ guarantees the recovery of the diffusive behaviour when $\eta \to 1^-$ by using the representation of the δ function

$$\delta(z-z_0) = \lim_{\epsilon \to 0} \epsilon |z-z_0|^{-(1\pm\epsilon)}. \quad (15)$$

Using this correlation function converts Eq. (11) into

$$\frac{\partial n_0}{\partial t} \simeq (\beta - 1) V_c^2 \tau_c^{2-\beta} \left(\int_0^t \frac{dt'}{(t-t')^{2-\beta}} \nabla^2 n_0(\mathbf{r},t') \right), \quad (16)$$

where we have defined the exponent $\beta \equiv 2 - \eta \in (1,2)$. The temporal integral can be easily rewritten as a fractional equation. Defining $\epsilon \equiv 1 - \beta \in (-1,0)$ we find

$$\int_0^t dt' \frac{G(t')}{(t-t')^{2-\beta}} = \int_0^t dt' \frac{G(t')}{(t-t')^{1+\epsilon}}$$
$$= \Gamma(-\epsilon)\, {}_0D^{\epsilon}{}_t G(t)$$
$$= \Gamma(\beta-1)\, {}_0D^{1-\beta}{}_t G(t), \quad (17)$$

where ${}_zD^a{}_t$ is the Riemann-Liouville FDO [40] of order a and start point at $t=z$. $\Gamma(x)$ is Euler's gamma function. Equation (16) then becomes

$$\frac{\partial n_0}{\partial t} \simeq {}_0D^{1-\beta}{}_t [D_\beta \nabla^2 n_0] \quad (18)$$

where $D_\beta \sim \Gamma(\beta) \tau_c^{2-\beta} V_c^2$ is the effective fractional diffusivity. Note that, since the order of the FDO is negative, the fractional operator in Eq. (18) is a fractional integral [40]. This equation exhibits superdiffusion, and it is known as the "fractional-time wave equation" because it becomes the usual wave equation when $\beta \to 2^-$ [47–50]. Interestingly, it cannot be recast in the form of Eq. (1), which precludes any relationship with some underlying separable CTRW based on stable distributions [51].

C. Subdiffusive case

The last choice we will discuss is again Eq. (14) but choosing instead $\eta \in (1,2)$. Note that the asymptotic tail is now negative, which implies anticorrelation at long times. It is often stated [6] that this choice leads to a subdiffusive equation formally identical to Eq. (18), but in which ${}_0D^{1-\beta}{}_t$ is a fractional differential operator since $\beta = 2 - \eta < 1$. In this case, one can introduce the Caputo temporal FDO of order β and start point 0, ${}^C_0D^{\beta}{}_t$, and transform Eq. (18) into the equivalent equation

$${}^C_0D^{\beta}{}_t n_0 \simeq D_\beta \nabla^2 n_0, \quad (19)$$

which is a particular realization of Eq. (1) with $\alpha = 2$. (This kind of equation has been obtained in many problems pertaining to plasma transport, for instance, for tracer transport in stochastic magnetic fields [8].)

The previous statement must, however, be taken with some care. The reason is that, to obtain Eq. (18) when $\beta < 1$, Eq. (16) must be integrated by parts:

$$\frac{\partial n_0}{\partial t} \simeq V_c^2 \tau_c^{2-\beta} \Bigg[-(t-t')^{-(1-\beta)} \frac{\partial^2 n_0}{\partial x^2}(x,t') \Bigg|_0^t + \int_0^t \frac{dt'}{(t-t')^{1-\beta}} \frac{\partial}{\partial t} \left(\frac{\partial^2 n_0}{\partial x^2}(x,t') \right) \Bigg]. \quad (20)$$

One can then obtain $\Gamma(\beta)\, {}_0D^{1-\beta}{}_t$ by combining the integral with the evaluation of the first term of the right-hand side at $t'=0$. But the remaining evaluation of the first term at $t'=t$ diverges. Equation (18) is thus obtained using Eq. (14) as correlation function [with $\eta \in (1,2)$] only if the divergent term can be neglected. Why and when can this be done?

To answer this question and illustrate the origin of the divergent term, we go back to Eq. (11) but using instead the model Lagrangian correlation [recall that $\eta \in (1,2)$]

$$C^L(\tau) = V_c^2 (\eta - 1) \left(1 + \frac{\tau}{\tau_c} \right)^{-2\eta} \left[1 - b \left(1 + \frac{\tau}{\tau_c} \right)^{\eta} \right]. \quad (21)$$

b will be chosen so that $b > 0$ and $|b| < 1$. $b > 0$ makes the asymptotic behavior of the model function, $C^L(\tau) \sim b V_c^2 \tau_c^{\eta} (1-\eta) \tau^{-\eta}$, be identical to that of Eq. (14).

$|b|<1$, on the other hand, provides it with the correct behavior (i.e., positive and finite) near the zero time lag, a fact that was ignored in Eq. (14). The neglect of the near-zero-lag behavior is, in fact, the reason why the divergent term appears for $\beta<1$.

Before inserting the model function into Eq. (11), it is worth computing the mean squared displacement it induces, which will give us a first hint at how and why the zero-lag behavior becomes important. Using Kubo's formula [46], given by

$$\langle \Delta x^2\rangle = 2\int_0^\tau d\tau'(\tau-\tau')C^L(\tau'), \tag{22}$$

it is easily found that (recall that $\beta=2-\eta$)

$$\langle \Delta x^2\rangle = \tilde{V}_c^2\tau_c\left(\frac{\tau_c^{3-2\beta}(\tau+\tau_c)^{2\beta-2}-\tau_c+2(1-\beta)\tau}{(3-2\beta)} + 2b\frac{\tau_c+\beta\tau-\tau_c^{1-\beta}(\tau+\tau_c)^\beta}{\beta}\right). \tag{23}$$

Thus, despite its negative (i.e., anticorrelated) tail, the model correlation means that for $\tau \gg \tau_c$

$$\langle \Delta x^2\rangle \sim 2\tilde{V}_c^2\tau_c\left(\frac{1-\beta}{3-2\beta}-b\right)\tau, \tag{24}$$

which is not subdiffusive, but diffusive. This contribution comes in fact from the evaluation of the Kubo integral at the lower limit $\tau=0$, where the near-zero-lag part of the correlation is important.

If we now insert the model correlation in Eq. (16) and carry out the same integration by parts as before, we obtain that, for $\tau \gg \tau_c$,

$$\frac{\partial n_0}{\partial t} = V_c^2\tau_c\left(\frac{1-\beta}{3-2\beta}-b\right)\frac{\partial^2 n_0}{\partial x^2}(x,t) + bV_c^2\tau_c^{2-\beta}\Gamma(\beta)D_t^{1-\beta}\left(\frac{\partial^2 n_0}{\partial x^2}(x,t)\right). \tag{25}$$

Comparing this result with Eq. (20), one finds that the divergent term has been substituted by the diffusive (and finite) term. The divergence is thus nonphysical and only due to the fact that the near-zero-lag behavior of the correlation function was neglected. However, one might think by looking at Eq. (25) that subdiffusion will never be observed in a turbulent system. This is not totally correct. A more careful inspection reveals that, for $b=b_0\equiv(1-\beta)/(3-2\beta)$, the diffusive term vanishes and subdiffusion dominates for all scales. Also, for $b=b_0\pm|\Delta b|$, the subdiffusive term in Eq. (24) dominates the transport scaling up to a time scale of the order of $\tau\sim\tau_c(b/\Delta b)^{1/(1-\beta)}$, which can in practice be significantly large. But it is only within this range of scales that it is justified to neglect the diffusive term (or, if we carry out the asymptotic calculation, to neglect the divergent term) and use Eq. (18) or (19) to describe transport.

D. Breakdown of the EQLR procedure

In this section we have reviewed previous results that show that the quasilinear renormalization of Eq. (4), in conjunction with the localization hypothesis, produces only FDEs that have a fractional derivative in time with exponent $\beta\in(0,2]$ and normal spatial derivatives with exponent $\alpha=2$. As a result, the transport exponent for these equations is always $H=\beta/2$, which gives subdiffusive transport for $\beta<1$, superdiffusive for $\beta>1$, and the classical diffusion equation for $\beta=1$. The range of dynamical behaviors captured by these equations is, however, quite restricted in α-β space: it is restricted to the $\alpha=2$ line in Fig. 1. And, in fact, all phenomenological models proposed for anomalous transport in plasmas use superdiffusive FDEs with $\alpha<2$ and $\beta<1$ [13–18,39], a case that is not included in Eq. (18). Why does EQLR fail to produce all these cases? Simply because the localization hypothesis cannot be justified whenever $\alpha<2$.

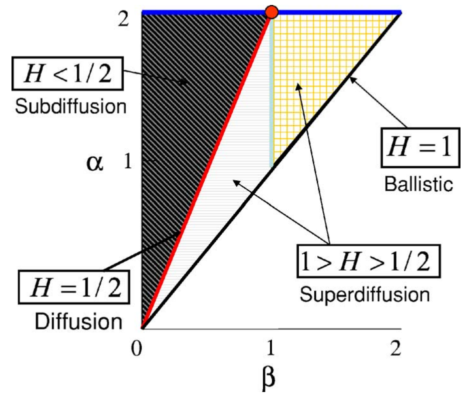

FIG. 1. (Color online) α-β parameter space for Eq. (1). Note that Eq. (18) refers only to the upper boundary ($\alpha=2$) of this space. The classical diffusive equation is simply a point at $\alpha=2$, $\beta=1$. The $\alpha=2$ line represents all the transport equations obtainable from EQLR.

III. BEYOND THE LAGRANGIAN VELOCITY CORRELATION FORMALISM: COMPUTATION OF THE ENSEMBLE AVERAGE IN EQ. (10)

It is now apparent that to renormalize Eq. (4) into an equation like Eq. (1) we must deal with the ensemble average that appears in Eq. (10) without invoking the locality hypothesis. The procedure described in this section has been developed to overcome this hurdle. We start from Eq. (10) but in its form prior to the $\mathbf{r}'$ integration. Next, we recast the ensemble average as an integral over the functional space Ω_V of all possible flow fields, weighted by some "probability measure on Ω_V," $d\mu(\tilde{\mathbf{V}})$:

$$\frac{\partial n_0}{\partial t} = \boldsymbol{\nabla}\cdot\left[\int_0^t dt'\int d\mathbf{r}'\boldsymbol{\nabla}n_0(\mathbf{r}',t')\times\left(\int_{\Omega_V} d\mu(\tilde{\mathbf{V}})\tilde{\mathbf{V}}(\mathbf{r},t)\tilde{\mathbf{V}}(\mathbf{r}',t')G(\mathbf{r},t|\mathbf{r}',t')\right)\right]. \tag{26}$$

The final objective is to show that, after some manipulations, the Ω_V integral can be identified with the kernels [40] that define the FDOs appearing in Eq. (1). The only assumptions we will rely on to achieve this objective must be derived from the fact that the flow is (homogeneous, isotropic) self-similar, in the sense that a well-defined transport exponent H exists. To evaluate the Ω_V integral, we first discretize the characteristic connecting $(\mathbf{r},t)$ to $(\mathbf{r}',t')$ in n spatial nodes. Then, assuming a constant time of flight between nodes ($\tilde{\mathbf{V}}_k$ denotes the velocity at the kth node), Eq. (9) can be integrated to yield the characteristic:

$$\mathbf{R}(t'|\mathbf{r},t) = \mathbf{r} - \frac{(t-t')}{n-1}\sum_{j=1}^{n-1}\tilde{\mathbf{V}}_j. \tag{27}$$

An n approximation of $G(\mathbf{r},t|\mathbf{r}',t')$ can then be obtained by inserting Eq. (27) in Eq. (8):

$$G(\mathbf{r},t|\mathbf{r}',t') = \delta\left(\mathbf{r}-\mathbf{r}' - \frac{(t-t')}{n-1}\sum_{j=1}^{n-1}\tilde{\mathbf{V}}_j\right). \tag{28}$$

Next, for $d\mu(\tilde{\mathbf{V}})$ we use

$$d\mu(\tilde{\mathbf{V}}) = \prod_{k=1}^{n} d\tilde{\mathbf{V}}_k P_n\left(\tilde{\mathbf{V}}_1,\ldots,\tilde{\mathbf{V}}_n\right), \tag{29}$$

where we have introduced P_n as the joint probability density function (PDF) of any *ordered* sequence of n node velocities actually corresponding to one realization of the flow field. P_n is a very complex function. But as the procedure advances, we will be able to reduce it by integrating out all the irrelevant intermediate node velocities. Then, all that is needed is to make physically based guesses for the reduced PDFs.

Next, by examining the propagator given by Eq. (28), it seems that some advantage may be gained by changing to new variables defined as

$$\hat{\mathbf{V}}_k \equiv \frac{1}{k}\sum_{j=1}^{k}\tilde{\mathbf{V}}_j, \quad k=1,\ldots,n. \tag{30}$$

Note, however, that the statistical properties of these variables would not be time independent in a self-similar flow. Indeed, since the tracer displacement must scale as $\langle|\Delta\mathbf{r}|^s\rangle \propto (t-t')^{sH}$ (for any real number $s>0$ for which this quantity converges), it happens that

$$\left\langle \left| \frac{1}{k}\sum_{j=1}^{k}\tilde{\mathbf{V}}_j \right|^s \right\rangle \sim (t-t')^{s(H-1)}. \tag{31}$$

This scaling will hold for any $k>n_{\min}$, where we can estimate $n_{\min}\sim n\tau_d/(t-t')$, τ_d being the decorrelation time of the Lagrangian velocity series. An important analytical advantage will be gained later by removing the time dependence in Eq. (31) at this point. To this effect, we now introduce new variables, referred to as "rescaled Lagrangian cumulative velocities" in what follows:

$$\bar{\mathbf{V}}_k \equiv (t-t')^{1-H}\left[\frac{1}{k}\sum_{j=1}^{k}\tilde{\mathbf{V}}_j\right], \quad k=1,\ldots,n. \tag{32}$$

It is also convenient to define a rescaled joint PDF,

$$\bar{P}_n(\{\bar{\mathbf{V}}_k\}) \equiv [n!\,(t-t')^{-n(1-H)}]^{n_D} P_n(\{\tilde{\mathbf{V}}_k\}), \tag{33}$$

which includes the Jacobian of the variable transformation. By moving to these rescaled Lagrangian cumulative velocities and introducing reduced PDFs defined as

$$\bar{P}_k^{[i_1,\ldots,i_k]}(\bar{\mathbf{V}}_{i_1},\ldots,\bar{\mathbf{V}}_{i_k}) \equiv \prod_{j\neq i_1,\cdots i_k}\int d\bar{\mathbf{V}}_j \bar{P}_n(\bar{\mathbf{V}}_1,\ldots,\bar{\mathbf{V}}_n), \tag{34}$$

we convert the integral over Ω_V appearing within large parentheses in Eq. (26) into

$$(t-t')^{2(H-1)}\int\int\int d\bar{\mathbf{V}}_1 d\bar{\mathbf{V}}_{n-1} d\bar{\mathbf{V}}_n \delta(\mathbf{r}'-\mathbf{r}+\bar{\mathbf{V}}_{n-1} \times (t-t')^H)\bar{\mathbf{V}}_1(n\bar{\mathbf{V}}_n-(n-1)\bar{\mathbf{V}}_{n-1})\bar{P}_3^{1,n-1,n}(\bar{\mathbf{V}}_1,\bar{\mathbf{V}}_{n-1},\bar{\mathbf{V}}_n). \tag{35}$$

Next, we eliminate the discretization variable by making $n\gg 1$. First, we introduce the conditional PDF

$$\bar{G}_3(\bar{\mathbf{V}}_n|\bar{\mathbf{V}}_1,\bar{\mathbf{V}}_{n-1}) \equiv \frac{\bar{P}_3^{1,n-1,n}(\bar{\mathbf{V}}_1,\bar{\mathbf{V}}_{n-1},\bar{\mathbf{V}}_n)}{\bar{P}_2^{1,n-1}(\bar{\mathbf{V}}_1,\bar{\mathbf{V}}_{n-1})}. \tag{36}$$

Since $\bar{\mathbf{V}}_{n-1}$ and $\bar{\mathbf{V}}_n$ coalesce to the same value at large n, one might be tempted to assume $G_3\simeq\delta(\bar{\mathbf{V}}_n-\bar{\mathbf{V}}_{n-1})$. This is correct up to first order in $1/n$ but, since $\bar{\mathbf{V}}_{n-1}$ and $\bar{\mathbf{V}}_n$ are multiplied by n (and $n-1$) in Eq. (35), we must go to the next order. Exploiting flow self-similarity again, it is straightforward to show that

$$\bar{G}_3(\bar{\mathbf{V}}_n|\bar{\mathbf{V}}_1,\bar{\mathbf{V}}_{n-1}) \simeq \delta\left(\bar{\mathbf{V}}_n - \frac{(n-1)+H}{n}\bar{\mathbf{V}}_{n-1}\right), \tag{37}$$

which is good up to order $1/n^2$. Using it, we integrate now Eq. (35) over $\bar{\mathbf{V}}_{n-1}$ and $\bar{\mathbf{V}}_n$ and get

$$\frac{H(\mathbf{r}-\mathbf{r}')}{(t-t')^{2+(n_d-1)H}}\int d\bar{\mathbf{V}}_1\bar{\mathbf{V}}_1\bar{P}_2^{1,n-1}\left(\bar{\mathbf{V}}_1,\frac{\mathbf{r}-\mathbf{r}'}{(t-t')^H}\right). \tag{38}$$

Instead of making an ansatz now for $\bar{P}_2$ that correlates $\bar{\mathbf{V}}_1$ and $\bar{\mathbf{V}}_{n-1}$, we introduce a second conditional probability

$$\bar{G}_2(\bar{\mathbf{V}}_k|\bar{\mathbf{V}}_{n-1}) \equiv \frac{\bar{P}_2^{k,n-1}(\bar{\mathbf{V}}_k,\bar{\mathbf{V}}_{n-1})}{\bar{P}_1^{n-1}(\bar{\mathbf{V}}_{n-1})}, \quad k\in[1,n], \tag{39}$$

so that Eq. (38) becomes $[(\mathbf{u}:\mathbf{v})_{ij}=u_i v_j$ represents the dyadic product$]$

$$\frac{H(\mathbf{r}-\mathbf{r}')}{(t-t')^{2+(n_d-1)H}}:\mathbf{\Phi}\left(\frac{\mathbf{r}-\mathbf{r}'}{(t-t')^H}\right)\bar{P}_1^{n-1}\left(\frac{\mathbf{r}-\mathbf{r}'}{(t-t')^H}\right), \tag{40}$$

where we have introduced the vector function

$$\mathbf{\Phi}(\bar{\mathbf{V}}_{n-1}) \equiv \int d\bar{\mathbf{V}}_1\bar{\mathbf{V}}_1 G_2(\bar{\mathbf{V}}_1|\bar{\mathbf{V}}_{n-1}). \tag{41}$$

We do not know what $\mathbf{\Phi}(\bar{\mathbf{V}}_{n-1})$ is, but if we average it over $\bar{\mathbf{V}}_{n-1}$, the result must be zero since

$$\int d\bar{\mathbf{V}}_{n-1}\mathbf{\Phi}(\bar{\mathbf{V}}_{n-1})\bar{P}_1(\bar{\mathbf{V}}_{n-1}) = \langle\bar{\mathbf{V}}_1\rangle = 0. \tag{42}$$

Then, since $\bar{P}_1(\bar{\mathbf{V}}_{n-1})$ is an even function, it is natural to assume that the vector $\mathbf{\Phi}(\bar{\mathbf{V}}_{n-1})$ is going to be odd in $\bar{\mathbf{V}}_{n-1}$. The natural choice is then to assume that it must be directed along $\bar{\mathbf{V}}_{n-1}$,

$$\mathbf{\Phi}(\bar{\mathbf{V}}_{n-1}) = g(\bar{\mathbf{V}}_{n-1})\bar{\mathbf{V}}_{n-1}, \tag{43}$$

with $g(\mathbf{A})$ being an even scalar function. Note that this is tantamount to saying that

$$G_2(\bar{\mathbf{V}}_1|\bar{\mathbf{V}}_{n-1}) = \Psi(\bar{\mathbf{V}}_1 - g(\bar{\mathbf{V}}_{n-1})\bar{\mathbf{V}}_{n-1}), \tag{44}$$

where the form of Ψ is unimportant at this level. This is easily proven by calculating

$$\mathbf{\Phi}(\bar{\mathbf{V}}_{n-1}) = \int d\bar{\mathbf{V}}_1\bar{\mathbf{V}}_1\Psi(\bar{\mathbf{V}}_1 - g(\bar{\mathbf{V}}_{n-1})\bar{\mathbf{V}}_{n-1})$$
$$= [Ag(\bar{\mathbf{V}}_{n-1})\bar{\mathbf{V}}_{n-1} + B], \tag{45}$$

where we have made the change of variables $\bar{\mathbf{V}}_1^* \equiv \bar{\mathbf{V}}_1 - g(\bar{\mathbf{V}}_{n-1})\bar{\mathbf{V}}_{n-1}$ and defined the two quantities

$$A \equiv \int d\bar{\mathbf{V}}_1^*\Psi(\bar{\mathbf{V}}_1^*), \quad B \equiv \int d\bar{\mathbf{V}}_1^*\bar{\mathbf{V}}_1^*\Psi(\bar{\mathbf{V}}_1^*). \tag{46}$$

It is easily proven that $A=1$ and $B=0$ *for any positive function* Ψ. This is a consequence of the fact that P_2 is a normalized and symmetric PDF. Indeed,

$$1 = \int d\bar{\mathbf{V}}_1 \int d\bar{\mathbf{V}}_{n-1}\bar{P}_2^{1,n-1}(\bar{\mathbf{V}}_1, \bar{\mathbf{V}}_{n-1})$$
$$= \int d\bar{\mathbf{V}}_1^*\Psi(\bar{\mathbf{V}}_1^*) = A, \tag{47}$$

$$0 = \int d\bar{\mathbf{V}}_1 \int d\bar{\mathbf{V}}_{n-1}\bar{\mathbf{V}}_1\bar{P}_2^{1,n-1}(\bar{\mathbf{V}}_1, \bar{\mathbf{V}}_{n-1})$$
$$= \int d\bar{\mathbf{V}}_1^*\bar{\mathbf{V}}_1^*\Psi(\bar{\mathbf{V}}_1^*) = B. \tag{48}$$

To complete the derivation, we must provide an expression for $g(\bar{\mathbf{V}}_{n-1})$. It is at this point that the benefit of using the rescaled Lagrangian cumulative velocities becomes apparent since, for a self-similar flow, we will argue that $g(\bar{\mathbf{V}}_{n-1})$ is a constant that depends solely on the characteristic exponents of the flow: $g(\bar{\mathbf{V}}_{n-1}) \simeq S(H, \ldots)$. We have checked this point by numerous tests with synthetic numerical data series as will be discussed later in Sec. IV. Therefore, using the form we just discussed for $\mathbf{\Phi}(\mathbf{V}_{n-1})$ in Eq. (40) and using that expression in Eq. (26) yields the equation that constitutes the central result of our formalism:

$$\frac{\partial n_0}{\partial t} \simeq HS(H, \ldots)\nabla \cdot \left[\int_0^t dt' \int d\mathbf{r}' \frac{|\mathbf{r}-\mathbf{r}'|^2}{(t-t')^{2+n_d H}} \times [\nabla n_0(\mathbf{r}', t') \cdot (\mathbf{u}_{\mathbf{r}-\mathbf{r}'}:\mathbf{u}_{\mathbf{r}-\mathbf{r}'})]\bar{P}_1^{n-1}\left(\frac{\mathbf{r}-\mathbf{r}'}{(t-t')^H}\right)\right]. \tag{49}$$

Note that Eq. (49) plays a role totally analogous to that previously played by Eq. (11) in the Lagrangian correlation formalism. [Note that the only reason why $\bar{P}_1^{n-1}(\bar{\mathbf{V}}_{n-1})$ appears instead of $\bar{P}_1^{n}(\bar{\mathbf{V}}_n)$ is because of having chosen a backward discretization formula in Eq. (27).] As a result, $\bar{P}_1^{n-1}(\bar{\mathbf{V}}_{n-1})$ and $S(H, \ldots)$ take over the role of the Lagrangian correlation matrix $\mathbf{C}^L$: They are the ones that must be provided by the physics of the problem in order to complete the renormalization.

How do we choose them? The first thing to realize is that $S(H, \ldots)$ will be a function of H plus any other exponent that characterizes $\bar{P}_1^{n-1}(\bar{\mathbf{V}}_{n-1})$. The simplest choice for the latter would then be some family of PDFs that introduces just one new exponent while still keeping some physical meaningfulness. One possibility is to use either a Gaussian or a symmetric Lévy PDF [52]. Lévy PDFs have the distinguishing feature of exhibiting power-law tails with exponent $-(\alpha + n_D)$, with $\alpha \in (0,2)$. Gaussians may be obtained from them in the limit $\alpha \to 2^-$. Note that this choice implies that the second characteristic exponent is $\alpha \in (0,2]$, thus converting $S(H, \ldots)$ into $S(H, \alpha)$, which we would have to determine for each choice. This choice has the added benefit of Gaussian and Lévy PDFs being the stable distributions predicted by the central limit theorem for the addition of uncorrelated, identically distributed variables [52]. One could argue that, in the case in which $H \neq 1/\alpha$, the Lagrangian velocities being added are not independent of each other. But we will stick to this choice even in that case, specially in the light of recent results that suggest that PDFs that behave asymptotically as Lévy PDFs are the limit distributions of sums of "self-similarly correlated" series as well.

IV. EXAMPLES: RENORMALIZATION OF THE ONE-DIMENSIONAL CASE

To illustrate the procedure just described, we will carry out explicitly the renormalization of the one-dimensional case. This might be thought to be a trivial exercise since $v = v(t)$ is the only divergence-free velocity field in one dimension. But it is easily shown that the same one-dimensional equation would describe transport in higher-dimensionality flows in which the mean density varies only along one preferred direction (say, the x axis). Then, one has that the dyadic term in Eq. (49) reduces to

$$\nabla n_0(\mathbf{r}', t') \cdot (\mathbf{r}-\mathbf{r}':\mathbf{r}-\mathbf{r}') = (\mathbf{r}-\mathbf{r}')\frac{\partial n_0}{\partial x}(x', t')(x - x'). \tag{50}$$

For our homogeneous and isotropic flow this causes the integrals over y' and z' appearing in Eq. (49) to vanish. As a result, we can consider $v_x(x,t)$ as the only dynamically meaningful component, while the remaining components of the velocity are responsible only for preserving the incompressibility of the flow.

With this in mind, we proceed to renormalize the one-dimensional version of Eq. (49) assuming $v=v(x,t)$. As we mentioned before, the two choices we will examine for $\bar{P}_1^{n-1}(\bar{V}_{n-1})$ are a Gaussian and a Lévy symmetric PDF:

A. Gaussian case: $\alpha=2$

To start, we consider the Gaussian case

$$\bar{P}_1^{n-1}(\bar{V}_{n-1}) \sim (4\pi\sigma_2)^{-1/2}\exp[-\bar{V}_{n-1}^2/(4\sigma_2)], \qquad (51)$$

where $\sigma_2>0$ is a constant. Then Eq. (49) becomes

$$\frac{\partial n_0}{\partial t} \simeq \frac{HS(H,2)}{\sqrt{4\pi\sigma_2}}\frac{\partial}{\partial x}\left[\int_0^t dt'\int_{-\infty}^{\infty} dx' \frac{|x-x'|^2}{(t-t')^{2+H}} \times\exp\left(-\frac{|x-x'|^2}{4\sigma_2(t-t')^{2H}}\right)\frac{\partial n_0}{\partial x}(x',t')\right]. \qquad (52)$$

Exploiting the fast decay of the exponential, we can Taylor expand the density gradient around x and keep the lowest order to get

$$\frac{\partial n_0}{\partial t} \simeq 2H\sigma_2 S(H,2)\left(\int_0^t \frac{dt'}{(t-t')^{2-2H}}\frac{\partial^2 n_0}{\partial x^2}(x,t')\right). \qquad (53)$$

At this point we can already say something about $S(H,2)$. Note that, in the limit $H\to 1/2^+$, Eq. (53) must reduce to the standard diffusive equation. To make this explicit, we rewrite $S(H,2)=p(H,2)(2H-1)$, where the function $p(H,2)$ is still to be determined. By writing it in this way, we can always use Eq. (15) to eliminate the integral in time while taking the diffusive limit.

Next, we deal with the time integral that appears in Eq. (53). Recalling that $\beta=2H$ for $\alpha=2$, this integral is then the same that we already discussed at length in Sec. II. We showed there that it can be rewritten as

$$\int_0^t \frac{dt' G(t')}{(t-t')^{2-2H}} \simeq \Gamma(2H-1)\; {}_0D^{1-\alpha H}{}_t G(t), \qquad (54)$$

where ${}_aD^{\sigma}{}_z$ represents the Riemann-Lioville FDO of order σ with respect to z and start point at $z=a$ [40]. Equation (54) is exact if $2H=\beta>1$ (which corresponds to the superdiffusive case). But if $2H=\beta<1$ (i.e., the subdiffusive case), it is valid only over the range of scales over which subdiffusion dominates the tracer transport. Keeping this restriction in mind in what follows, we proceed further and rewrite Eq. (53) as

$$\frac{\partial n_0}{\partial t}(x,t) \simeq {}_0D_t^{1-2H}\left(D_{[H,2]}\frac{\partial^2 n_0}{\partial x^2}(x,t)\right), \qquad (55)$$

with fractional diffusivity $D_{[H,2]}\equiv\sigma_2 p(H,2)\Gamma(2H+1)$.

Before discussing $p(H,2)$, note that Eq. (55) is identical to Eq. (18), which was obtained with the Lagrangian correlation formalism. If $2H=\beta<1$, one could introduce a Caputo FDO of order β and rewrite Eq. (55) in the form of Eq. (1). If $2H=\beta>1$ (superdiffusive case), Eq. (55) becomes again the fractional-time wave equation [47–50] that was also obtained from the Lagrangian correlation formalism [see Eq. (18)].

There is not much more we can say about $p(H,2)$ from the fluid limit, except for the fact that $p(1/2,2)=1$ so that $D_{1/2,2}\to\sigma_2$, the diffusive result. But we can determine it by noting that the Lagrangian series of velocities must be (on average) a fractional Brownian noise (FBN) series of Hurst exponent H for the PDF of the cumulative Lagrangian velocities to be given by Eq. (51). The long-time behavior of the correlation function of a FBN series $\{v_k, k=1,2,\dots\}$ is known and equal to [52]

$$\langle v_j v_k\rangle \sim 2H(2H-1)\sigma|j-k|^{2H-2}, \qquad (56)$$

σ being the variance of the series. If we compute the same correlation for the Lagrangian velocities of the flow using the techniques introduced in this paper we obtain

$$\langle V(x,t)V(X(t'|x,t))\rangle \sim 2H(2H-1)p(H,2)\sigma_2(t-t')^{2H-2}, \qquad (57)$$

from which we can conclude that $p(H,2)=1$.

We proceed now to describe some numerical evidence from synthetic fBn signals gathered to support the validity of the two hypotheses on which the previous calculations stand [i.e., that $g(\mathbf{V}_{n-1})$ is constant and that $p(H,2)=1$]. We have generated synthetic FBN series $\{v_i,\ i=1,\dots,M\}$ with prescribed exponent H between 0 and 1 (within a 5% error) and used them as a (discrete) realization of the velocity Lagrangian time series. Then, we have formed the associated rescaled Lagrangian cumulative velocity series $\{\bar{v}_i,\ i=1,\dots,N\}$ by calculating the sum

$$\bar{v}_j = \frac{1}{n^H}\sum_{j=1}^{n} v_j, \qquad (58)$$

using a sliding window of size $n\ll M$ so that the original and the rescaled Lagrangian cumulative velocity series have equal length. Some contour levels of the conditional PDF $G_2(n^{1-H}v_1|\bar{v})$ are shown in Fig. 2 (remember that $\bar{v}_1=n^{1-H}v_1$). Clearly, the linear relationship that we assumed between $\bar{\mathbf{V}}_1$ and $\bar{\mathbf{V}}_{n-1}$ in Eq. (44) after setting $g(\mathbf{V}_{n-1})=S(H,2)$ is verified. And the slope is well fitted with $S(H,2)=2H-1$, which confirms that $p(2,H)=1$ (see Fig. 3). [Note that a small deviation from this law is found for the anticorrelated signals ($H<1/2$), but this behavior is due to the dominance of the diffusion associated with the central peak of the autocorrelation function of the numerical series over subdiffusion, as we already discussed in Sec. II.]

B. Lévy case: $\alpha<2$

Next, we consider a PDF that behaves asymptotically like a symmetric Lévy PDF with index $0<\alpha<2$ [52]:

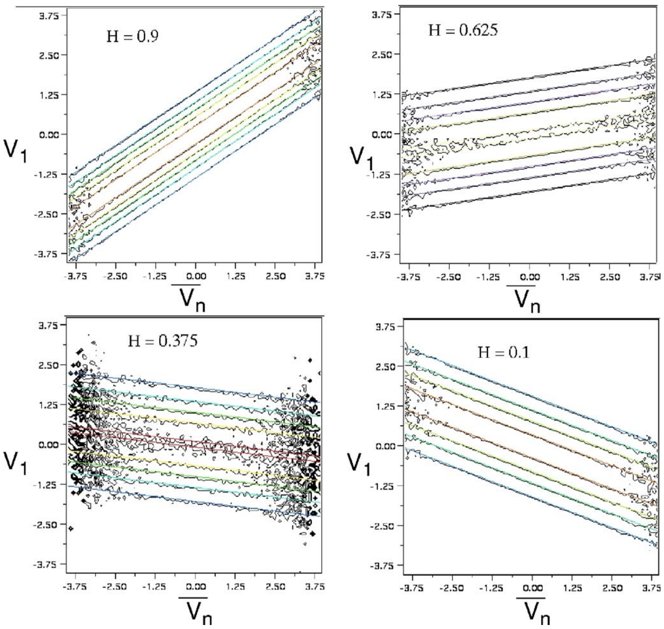

FIG. 2. (Color online) Upper left: $G_2(n^{1-H}v_1|\bar{v}_n)$ for $H=0.9$; upper right: $G_2(n^{1-H}v_1|\bar{v}_n)$ for $H=0.625$; lower left: $G_2(n^{1-H}v_1|\bar{v}_n)$ for $H=0.375$; lower right: $G_2(n^{1-H}v_1|\bar{v}_n)$ for $H=0.1$.

$$\bar{P}_1^{n-1}(\bar{V}_{n-1}) \sim \frac{\sigma_\alpha(1-\alpha)\alpha}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)}|\bar{V}_{n-1}|^{-(1+\alpha)}, \quad (59)$$

where $\sigma_\alpha > 0$ is a constant. If we insert it into Eq. (49) it is easily obtained that

$$\frac{\partial n_0}{\partial t} \simeq -\frac{\sigma_\alpha \alpha H(\alpha-1)S(H,\alpha)}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)}\int_0^t \frac{dt'}{(t-t')^{2-\alpha H}} \times \frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty}\frac{dx'}{|x-x'|^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t')\right). \quad (60)$$

Similarly to what we did in the Gaussian case, we force the correct diffusive limit by rewriting

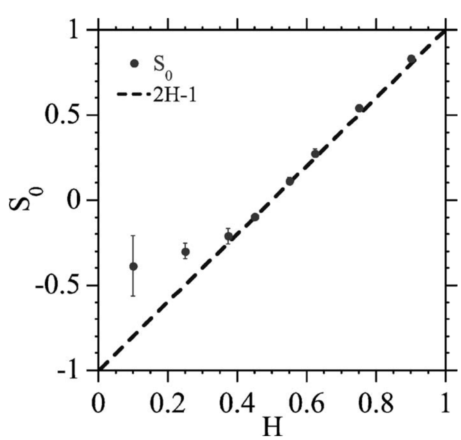

FIG. 3. Graph of $S(H,2)$ vs H for several synthetic FBN signals.

$$S(H,\alpha) = p(H,\alpha)\left(\frac{\alpha H-1}{\alpha-1}\right). \quad (61)$$

In this way, the spatial and temporal integrals will vanish when using again the identity Eq. (15) in the diffusive limit (i.e., $\alpha \to 2^-$, $H \to 1/2^+$). [For the spatial one, we need to use that $\Gamma(x) \simeq x^{-1}$ around the origin.] $p(H,\alpha)$ is again an unknown function that must still be determined. Using this expression for $S(H,\alpha)$ we rewrite Eq. (60) as

$$\frac{\partial n_0}{\partial t} \simeq -\frac{\sigma_\alpha \alpha H(\alpha H-1)p(H,\alpha)}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)}\int_0^t \frac{dt'}{(t-t')^{2-\alpha H}} \times \frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty}\frac{dx'}{|x-x'|^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t')\right). \quad (62)$$

Straightforward algebra then shows that (see the Appendix), for any $\alpha \in (0,2]$,

$$\frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty}\frac{dx'}{|x-x'|^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t')\right) = -2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)\frac{\partial^\alpha}{\partial|x|^\alpha}, \quad (63)$$

where $\partial^\alpha/\partial|x|^\alpha$ is the Riesz symmetric FDO of order α [9]. On the other hand, for $\alpha<2$ it happens that $\beta=\alpha H$, and the temporal integral is again the same that we already have discussed in the Gaussian case [Eq. (54)]. Keeping in mind the restrictions discussed then for the $\beta<1$ case (which, for $\alpha<2$, may be not subdiffusive), we proceed to combine those results with Eq. (63) and use them to rewrite Eq. (62) in the form

$$\frac{\partial n_0}{\partial t}(x,t) \simeq {}_0D_t^{1-\alpha H}\left(D_{[H,\alpha]}\frac{\partial^\alpha}{\partial|x|^\alpha}n_0(x,t)\right). \quad (64)$$

$D_{[H,\alpha]} \equiv \sigma_\alpha p(H,\alpha)\Gamma(\alpha H+1)$ is the fractional diffusivity. This is the final result of the renormalization. Note that, if $\beta \to 1^-, \alpha \to 2^-$, Eq. (64) becomes the usual diffusive equation, as it should. Also, if we take only the limit $\alpha \to 2^-$ but leave β (or H) arbitrary, Eq. (64) becomes the final result produced by EQLR [i.e., Eq. (18)]. For arbitrary α and $\beta=\alpha H$, subdiffusive behavior is exhibited whenever $2\beta<\alpha$, diffusive if $2\beta=\alpha$, and superdiffusive if $2\beta>\alpha$. For $\beta<1$, it can be recast into the more familiar form

$${}_0^C D_t^\beta n_0(x,t) = D_{[H,\alpha]}\frac{\partial^\alpha}{\partial|x|^\alpha}n_0(x,t), \quad (65)$$

which is identical to Eq. (1) and whose derivation was the main motivation for this work. On the other hand, Eq. (64) provides a Lévy version of the aforementioned fractional-time wave equation [47,48] for $\beta>1$, which also reduces to the standard wave equation if $\alpha \to 2^-$, $\beta \to 2^-$.

Regarding the value of $p(H,\alpha)$ we find again that, by enforcing the appropriate diffusive limit, the only thing that can be said is that $p(1/2,2)=1$. We have relied now on numerical evidence alone to estimate its value. We have repeated the numerical experiments we did for the FBN, but using instead synthetic series of Lévy, H-correlated noise

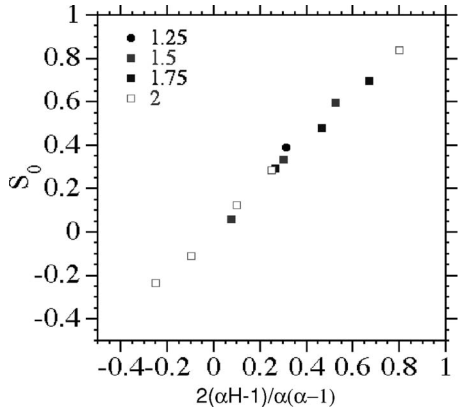

FIG. 4. Graph of $S(H,\alpha)$ vs $2(\alpha H-1)/\alpha(\alpha-1)$ for synthetic Lévy correlated signals with $H>1/\alpha$ and $\alpha\in[1,2]$.

[52]. In this case, the anticorrelated series become dominated by the diffusive behavior of the central peak of the autocorrelation even more dramatically than we already observed for the FBN signals. But if we stick to the correlated ones, we find that the numerical results are consistent with Eq. (61) and give $p(H,\alpha)\simeq 2/\alpha$ within numerical errors (see Fig. 4).

V. CONCLUSIONS

In summary, we have proven that incompressible turbulent flows that satisfy quite general self-similarity conditions can be renormalized into a much wider variety of transport equations than those that the Lagrangian correlation formalism allows. The key is to avoid the use of the localization hypothesis, for which we have provided a procedure based on applying functional integration techniques to the fluctuating tracer trajectories.

In the one-dimensional case, it is found that the renormalization scheme reduces to the usual fractional differential equations under quite general conditions. The fractional order of the resulting transport FDE depends on two exponents H and α, which are respectively related to the degree of correlation of the Lagrangian velocity series (i.e., its Hurst exponent) and to the exponent of the asymptotic tail of the PDF of the rescaled Lagrangian cumulative velocities. These two exponents can be combined to define a third exponent $\beta\equiv\alpha H$, which determines whether the FDE is Markovian ($\beta=1$) or not ($\beta\neq 1$). When $\beta\leq 1$, the linear FDE obtained in this way coincides with the fluid limits of (microscopic) separable CTRWs based on stable distributions for the waiting times and step sizes of the microscopic walkers [2–5]. This is an important connection since, in the diffusive case ($\beta=1,\alpha=2$), it ultimately justifies estimating transport coefficients as the square of a typical step size divided by the typical waiting time. However, when $\beta>1$, the resulting FDEs are Lévy generalizations of the fractional-time wave equation [47,48], which are not the fluid limits of any separable CTRW based on stable distributions [51]. This is an important result in itself, since it implies the interpretation of turbulent transport in terms of a stochastic random walk—with average step size provided by the typical eddy size and average waiting time by the eddy turnover time—may not be always the most appropriate.

The renormalization procedure presented here also provides us with a first step toward establishing a more rigorous formal basis for the many phenomenological CTRW and FDE models recently proposed to describe turbulent transport in magnetically confined plasmas which have $\alpha<2$ [13–18,39]. It also suggests that the analysis of the scaling properties of the rescaled Lagrangian cumulative velocities introduced here may provide another very useful tool to "measure" the relevant fractional exponents, but the implementation of these ideas will be explored in a future publication. Fractional models built using these exponents may capture more efficiently the nondiffusive features observed in fusion experiments and the large-scale turbulence codes that are playing a dominant role in controlled thermonuclear fusion research [25,26]. But we remark again that the applications of these results are not restricted to plasma transport. The formalism presented here may be relevant in other fields where a turbulent flow is present: fluid or atmospheric turbulence, combustion, and others.

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APPENDIX: DERIVATION OF EQ. (63)

To prove Eq. (63), the integral inside parentheses in the left-hand side of the equation must be first rewritten as

$$\int_{-\infty}^{\infty}\frac{dx'}{|x-x'|^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t')=\int_{-\infty}^{x}\frac{dx'}{(x-x')^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t') - \int_{x}^{\infty}\frac{dx'}{(x'-x)^{\alpha-1}}\left(\frac{\partial n_0}{\partial(-x)}(x',t')\right). \tag{A1}$$

In the case $\alpha\in(1,2)$, the two terms correspond to the spatial Caputo derivatives of order $(\alpha-1)\in[0,1]$, respectively, with start point at $-\infty$ and end point at ∞, except for a factor $\Gamma(2-\alpha)$. Moreover, since n_0 vanishes at both upper and lower limits, the Caputo derivatives can be identified with their Riemann-Liouville counterparts:

$$\frac{\partial}{\partial x}\left[\int_{-\infty}^{\infty}\frac{dx'}{|x-x'|^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t)\right]$$
$$=\Gamma(2-\alpha)\frac{\partial}{\partial x}\left[{}_{-\infty}D_x^{\alpha-1}n(x,t)-{}^{\infty}D_x^{\alpha-1}n(x,t)\right]$$
$$=\Gamma(2-\alpha)\left[{}_{-\infty}D_x^{\alpha}n(x,t)+{}^{\infty}D^{\alpha}{}_x n(x,t)\right]$$
$$=-2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)\frac{\partial^\alpha}{\partial|x|^\alpha}, \qquad (A2)$$

where we have introduced the Riesz FDO and used the fact that, by acting on the left-hand side with $\pm\partial/\partial(\pm x)$, we can increase the order of the fractional derivatives.

In the case $\alpha\in(0,1)$, we need to integrate by parts each of the two integrals in the right-hand side of Eq. (A1). The first one becomes

$$\int_{-\infty}^{x}\frac{dx'}{(x-x')^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t')=(1-\alpha)\int_{-\infty}^{x}\frac{dx'}{(x-x')^{\alpha}}n_0(x',t') \qquad (A3)$$

and a similar expression can be obtained for the second one. Combining these two expressions and acting on them with the divergence operator we can write that

$$\frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty}\frac{dx'}{|x-x'|^{\alpha-1}}\frac{\partial n_0}{\partial x}(x',t)\right)$$
$$=\Gamma(2-\alpha)\frac{\partial}{\partial x}\left[{}_{-\infty}D^{\alpha-1}{}_x n(x,t)-{}^{\infty}D^{\alpha-1}{}_x n(x,t)\right]$$
$$=-2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)\frac{\partial^\alpha}{\partial|x|^\alpha}. \qquad (A4)$$

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